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# Generalized scalar particle quantization in 1 + 1 dimensions and $D(2, 1; \alpha)$

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**Abstract**

The exceptional superalgebra $D(2, 1; \alpha)$ has been classified as a candidate conformal supersymmetry algebra in two dimensions. We propose an alternative interpretation of it as an extended BFV–BRST quantization superalgebra in $2D$ ($D(2, 1; 1) \simeq osp(2, 2|2)$). A superfield realization is presented wherein the standard extended phase space coordinates can be identified. The physical states are studied via the cohomology of the BRST operator. Finally we reverse engineer a classical action corresponding to the algebraic model we have constructed, and identify the Lagrangian equations of motion.

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## 1. Introduction and main results

In two previous papers we have examined the covariant BFV–BRST quantization of the scalar [1] and spinning [2] particle respectively. In both these papers we started with a physical model for the system, whose quantization was shown to admit the real Lie superalgebra $iosp(d, 2/2)$ as an underlying symmetry. In this paper we take an algebraic approach; as $osp(d, 2/2)$ is a member of the class of classical simple Lie superalgebras, by an appropriate generalization it should be possible to extend the quantization superalgebra $iosp(d, 2/2)$ into a more general classical simple Lie superalgebra. The motivation behind this is the need for a characterization of admissible spacetime 'BFV–BRST extended' supersymmetries in various dimensions. Here we demonstrate this by studying the particular case of $d = 2$, which leads to the quantization of two-dimensional relativistic particles in the exceptional superalgebra $D(2, 1; \alpha)$.

In this section we briefly define and review the properties of the exceptional superalgebra $D(2, 1; \alpha)$. In section 2 we shall construct superfield representations of the BFV–BRST quantization superalgebra corresponding to $D(2, 1; \alpha)$ and study the physical states via the

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BRST operator. This will be done using only the algebraic structure as a guide (i.e. no physical model). Finally, in section 3 we shall reverse engineer a classical action corresponding to the algebraic model we have constructed, and identify the corresponding Lagrangian equations of motion. A preliminary version of the results contained in this paper was published in [3].

The classical simple Lie superalgebras consist of the $spl(m/n)$ and the $osp(m/2n)$ families, the strange series $P(n)$ and $Q(n)$ and the exceptional algebras $F(4)$, $G(3)$ and $D(2,1;\alpha)$. Comprehensive definitions and descriptions of these algebras can be found in several places, see for example [4–7]. A study of the $D(2,1;\alpha)$ algebras, including a detailed analysis of their finite- and infinite-dimensional irreducible representations, has been carried out by Van der Jeugt [8]. The explicit supercommutation relations of the $D(2,1;\alpha)$ superalgebras are given in [9].

The algebras $D(2,1;\alpha)$ are a one-parameter family of 17-dimensional non-isomorphic Lie superalgebras. For the special case of $\alpha = 1$ we have $D(2,1;1) \cong D(2,1) \cong osp(4,2)$. It is through this special case that we seek to generalize the BFV–BRST quantization algebra. This aspect will be discussed in more detail in the next section.

The even part of the real superalgebra $D(2,1;\alpha)$ is the nine-dimensional non-compact form $sl(2,\mathbb{R}) + sl(2,\mathbb{R}) + sl(2,\mathbb{R})$, whilst the odd part (of dimension eight) is the spinorial representation $(2,2,2)$ of the even part. The parameter $\alpha$ appears only in the anti-commutation relations among the components of the tensor products (i.e. the odd components). In terms of the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1^2 = -(1+\alpha)/2$, $\varepsilon_2^2 = 1/2$, $\varepsilon_3^2 = \alpha/2$ and $\varepsilon_i \cdot \varepsilon_j = 0$ if $i \neq j$, the root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \{\pm 2\varepsilon_i\} \qquad \text{and} \qquad \Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}\,.$$

In [10], Günaydin studied $D(2,1;\alpha)$ considered as the superconformal symmetry group of an family of exotic superspaces in two dimensions defined by the one-parameter family of Jordan superalgebras $JD(2/2)_\alpha$. In this paper he also derived the full super-differential operators representing the actions of $D(2,1;\alpha)$ on the exotic superspaces. Here we similarly derive a superfield realization; however, it is over a different superspace.

## 2. The quantized $D(2,1;\alpha)$ particle

### 2.1. Preliminaries

The BFV–BRST quantization of relativistic systems provides a cohomological resolution of irreducible unitary representations (unirreps) of spacetime symmetries. Moreover these unirreps appear to be associated with constructions of $iosp(d,2/2)$ for relativistic particles in flat spacetime, as can be seen in [1] and [2]. In this paper, however, we do not invoke translations as additional generators and so the algebra reduces to $osp(d,2/2)$. Here we follow an algebraic approach, and so require a classification of admissible 'quantization superalgebras' in various dimensions. Some examples of such algebras for Minkowski space $(d-1,1)$ [10, 11] are $D(2,1;\alpha)$ in $d=2$ (note that $\alpha=1$ corresponds to $osp(2,2/2)$); in $d=3$ we have $osp(3,2/2)$ which corresponds to anti-de Sitter symmetry (which may thus be relevant to anyon quantization), and for $d = 3+1$ we get conformal symmetry of four-dimensional spacetime, and super unitary superalgebras, as possible alternative quantization superalgebras.

In order to detail our construction for the $d=2$ case, we firstly outline the scalar particle quantization in generic $d$. In second-order form [12] the action is

$$S = m\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \sqrt{\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\eta_{\mu\nu}} \qquad (1)$$

which leads to the canonical momenta

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}.$$

The system has one constraint: the mass shell condition

$$\phi_1 = p^\mu p_\mu - m^2 = 0.$$

In accordance with the BRST prescription, we enlarge the phase space by treating the Lagrange multiplier λ, associated with this constraint, as a dynamical variable. We then introduce a vanishing conjugate momentum $\pi_\lambda$ associated with λ, which forms a second constraint on the system. The Poisson brackets are the usual ones: $\{p_\mu, x^\nu\} = \delta_\mu^\nu$, $\{\lambda, \pi_\lambda\} = 1$.

The extended action (1) is invariant under the following infinitesimal gauge transformations:

$$\begin{aligned}
\delta\lambda &= \dot{\varepsilon} \\
\delta x^\mu &= \{x^\mu, \varepsilon\phi_1\} = 2\varepsilon p^\mu \\
\delta p^\mu &= \{p^\mu, \varepsilon\phi_1\} = 0
\end{aligned}$$

with $\varepsilon(\tau)$ being an arbitrary (dimensionless) infinitesimal function such that $\varepsilon(\tau_i) = \varepsilon(\tau_f) = 0$.

As in our two previous papers [1, 2] it is necessary to place two restrictions on the system so that the quantum formulation should be consistent with the equations of motion and guage fixing at the classical level. Firstly, we choose to work in the class [13–16] $\dot{\lambda} = 0$; moreover we take gauge fixing with respect to gauge transformations in one of the connected components of the group. This is equivalent to limiting the quantization of λ to the half line ($\mathbb{R}^+$ or $\mathbb{R}^-$). As a results λ will be quantized on the half line and the system is not modular invariant until the two distinctly oriented sectors (particle and anti-particle) are combined. Secondly, we take as a first-class constraint $\phi_2 = \lambda\pi_\lambda$ (rather than the usual $\phi_2 = \pi_\lambda$ used in the standard construction). Note that $\phi_2 = \lambda\pi_\lambda$ is regular in the sense of Govaerts [12] provided $\lambda \neq 0$.

The BFV extended phase space [12, 19] for the BRST quantized $D(2, 1; \alpha)$ particle can now completed by introducing the Grassmann odd conjugate pairs of ghosts $\eta^1$, $\rho_1$ and $\eta^2$, $\rho_2$ corresponding to the constraints $\phi_1$ and $\phi_2$ respectively. Thus the full phase space comprises the canonical variables

$$x^\mu(\tau), p_\mu(\tau), \lambda(\tau), \pi_\lambda(\tau), \eta^{1,2}, \rho_{1,2}.$$

This allows us to define the canonical BRST operator as

$$\Omega = \eta^1\phi_1 + \eta^2\phi_2.$$

With a gauge fixing function [13] defined as $\mathcal{F} = -\frac{1}{2}\lambda\rho_2$ the Hamiltonian can be calculated as

$$H = \{\mathcal{F}, \Omega\} = -\tfrac{1}{2}\lambda\left(\eta^1\rho_2 + p^\mu p_\mu - m^2\right). \tag{2}$$

In quantizing the system via the standard Schrödinger representation we have the operators $X^\mu$, $P_\nu$ corresponding to $x^\mu$, $p_\nu$ acting on suitable sets of functions $x^\mu$ on the half line. For the remaining operators $(\lambda(\tau), \pi_\lambda(\tau), \eta^{1,2}, \rho_{1,2})$ we find it convenient to define [17] $\theta_\alpha$ and $Q_\alpha$ $(\alpha = 1, 2)$ by

$$Q_{1,2} = \frac{1}{2\sqrt{2}}\left(2\eta^1 \pm \rho_2\right) \qquad \theta_{1,2} = \frac{1}{\sqrt{2}}\left(\pm\rho_1 - 2\eta^2\right)$$

and $X_- = 1/2(\lambda\phi_\lambda + \phi_\lambda\lambda)$, $P_+ = \lambda^{-1}$, along with $X_+ = \tau$ and $P_- = H$, where $H$ is the Hamiltonian. Thus the full set of operators in the Schrödinger representation is

$$X^\mu, P_\nu \qquad \lambda, \pi_\lambda \qquad Q_\alpha, X_\beta \qquad P_+, X_- \qquad P_-, X_+. \tag{3}$$

The non-zero commutation relations between these operators are

$$\begin{array}{lll} \left[X_\mu, P_\nu\right] = -\mathrm{i}\eta_{\mu\nu} & \left\{X_\alpha, Q_\beta\right\} = \mathrm{i}\varepsilon_{\alpha\beta} & \left[X_-, P_+\right] = \mathrm{i} \\ \left[X_-, P_-\right] = -\mathrm{i}P_+^{-1}P_- & \left[X_\alpha, P_-\right] = \mathrm{i}P_+^{-1}Q_\alpha & \left[X_\mu, P_-\right] = \mathrm{i}P_+^{-1}P_\mu. \end{array} \tag{4}$$

Having set up our system as such, we find that the operators $J_{AB}$ defined as

$$\begin{array}{ll} J_{\mu-} = X_\mu P_- - X_- P_\mu & J_{\mu\alpha} = L_{\mu\alpha} = X_\mu Q_\alpha - \theta_\alpha P_\mu \\ J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu & J_{+\mu} = X_+ P_\mu - X_\mu P_+ \\ J_{+\alpha} = L_{+\alpha} = X_+ Q_\alpha - \theta_\alpha P_+ & J_{\alpha\beta} = K_{\alpha\beta} = \theta_\alpha Q_\beta + \theta_\beta Q_\alpha \\ J_{+-} = X_- P_+ - X_+ P_- & J_{\alpha-} = L_{\alpha-} = \theta_\alpha P_- - X_- Q_\alpha \end{array} \tag{5}$$

generate the inhomogeneous orthosymplectic superalgebra $iosp(d, 2/2)$. Note that in (5) we have used the symbol $J$ to represent elements of the even subalgebra $so(d, 2)$, $K$ to represent the elements of the even subalgebra $sp(2, \mathbb{R})$ and $L$ the remaining odd generators. The graded commutation relations of this algebra were given in [18], and are repeated here as

$$\begin{aligned} [\![J_{MN}, J_{PQ}]\!] = \mathrm{i}(&\eta_{NQ} J_{MP} - [NP]\eta_{NP} J_{MQ} \\ &-[MN][MP]\eta_{MP} J_{NQ} + [PQ][MN][MQ]\eta_{MQ} J_{NP}). \end{aligned} \tag{6}$$

$$[\![J_{MN}, P_L]\!] = \mathrm{i}(\eta_{LN} P_M - [MN]\eta_{LM} P_N) \tag{7}$$

where the sign factors $[MN] \equiv (-1)^{mn}$ are $-1$ when both indices are fermionic.

In the $d = 2$ case, in order to extend $osp(2, 2/2)$ to $D(2, 1; \alpha)$, we must modify three of the anticommutation relations given in (6) and (7) (with the rest remaining the same). The new relations are

$$\begin{aligned} &\{L_{\mu\alpha}, L_{\nu\beta}\} = \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}(J + AJ_{+-}) - \eta_{\mu\nu}K_{\alpha\beta} \\ &\{L_{\mu\alpha}, L_{\beta\pm}\} = -\varepsilon_{\alpha\beta}(J_{\mu\pm} \pm B_\pm \varepsilon_\mu^{\ \nu} J_{\nu\pm}) \\ &\{L_{\alpha\pm}, L_{\beta\mp}\} = \pm\varepsilon_{\alpha\beta}(J_{+-} \pm C_\pm J) - K_{\alpha\beta}. \end{aligned} \tag{8}$$

For simplicity, we recognize that $J_{\mu\nu}$ is anti-symmetric. This allows us to define the operator $J$ by

$$J_{\mu\nu} = \varepsilon_{\mu\nu} J. \tag{9}$$

Taking the super-Jacobi identity on $L_{\mu\alpha}$, $L_{\nu\beta}$ and $L_{\gamma\pm}$ it is straightforward to show that

$$B_\pm = \frac{\mp A}{\det(\eta)}.$$

Taking the super-Jacobi identity on $L_{\mu\alpha}$, $L_{\mu\pm}$ and $L_{\gamma\mp}$ we can show

$$C_\pm = \frac{-A}{\det(\eta)}.$$

Through the use of Cartan generators and weight operators we can eliminate all but one of $A$, $B_\pm$, $C_\pm$ and relate them back to the $\alpha$ parameter in $D(2, 1; \alpha)$. This results in new generators

$$\begin{aligned} &\tilde{J} = J + aJ_{+-} \\ &\tilde{J}_{+-} = J_{+-} + aJ \end{aligned} \tag{10}$$

with the single parameter $a$, given by

$$a = \frac{1-\alpha}{1+\alpha}. \tag{11}$$

The modified anti-commutation relations (8) can now be written

$$\begin{aligned} &\{L_{\mu\alpha}, L_{\nu\beta}\} = \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta} \\ &\{L_{\mu\alpha}, L_{\beta\pm}\} = -\varepsilon_{\alpha\beta}(J_{\mu\pm} \pm a\varepsilon_\mu^{\ \nu} J_{\nu\pm}) \\ &\{L_{\alpha\pm}, L_{\beta\mp}\} = \pm\varepsilon_{\alpha\beta}\tilde{J}_{+-} - K_{\alpha\beta}. \end{aligned} \tag{12}$$

Note that the odd generators close compactly on $\tilde{J}$, $\tilde{J}_{+-}$ (although at the expense of more complicated commutation brackets). The invariant bilinear form on $D(2,1;\alpha)$ is

$$\begin{aligned}
&(J,J)=1\\
&(J,J_{+-})=(J_{+-},J)=-a\\
&(J_{+-},J_{+-})=1\\
&(J_{\mu\pm},J_{\nu\pm})=-\eta_{\mu\nu}\pm a\varepsilon_{\mu\nu}\\
&(K_{\alpha\beta},K_{\gamma\delta})=(1-a^2)\left(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}+\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}\right)\\
&(L_{\mu\alpha},L_{\nu\beta})=(1-a^2)\eta_{\mu\nu}\varepsilon_{\alpha\beta}\\
&(L_{\alpha\pm},L_{\beta\mp})=(1-a^2)\varepsilon_{\alpha\beta}.
\end{aligned}$$

And the Casimir $C$ can be written

$$C=C_1+C_2 \tag{13}$$

where

$$\begin{aligned}
&C_1=-J^2-J_{+-}^2+\left\{J_+^\mu,J_{\mu-}\right\}-\tfrac{1}{2}K^{\alpha\beta}K_{\alpha\beta}-L^{\mu\alpha}L_{\mu\alpha}-\left[L_+^\alpha,L_{\alpha-}\right]\\
&C_2=-a\left\{J,J_{+-}\right\}-a\varepsilon^{\mu\nu}\left\{J_{\mu+},J_{\nu-}\right\}.
\end{aligned}$$

### 2.2. *Superfield realization*

For the $D(2,1;\alpha)$ particle we do not have a physical model such as that laid out between equations (1) to (3), and so must use the algebraic structure as our only guide. We regard $D(2,1;\alpha)$ as a generalization of $osp(d,2/2)$ (for the $d=2$ case) and seek to find a superfield realization which is equivalent to the case for the scalar relativistic particle for $\alpha=1$ and $d=2$.

In the generic $d$-dimensional $osp(d,2/2)$ case we can define the homogeneous manifold

$$\mathcal{M}=OSp(d,2/2)/G_0$$

where the stability group $G_0$ is the semi-direct product

$$G_0=OSp(d-1,2/2)\wedge\mathcal{N}$$

and

$$\begin{aligned}
&OSp(d-1,2/2)=\langle J_{\mu\nu},L_{\mu\alpha},K_{\alpha\beta}\rangle\\
&\mathcal{N}=\langle J_{\mu-},L_{\alpha-}\rangle.
\end{aligned} \tag{14}$$

For one-parameter subgroups $g(t)$ with generator $A$, the standard superfield realization leads to generators $\hat{A}$ acting on functions $\phi$ over $\mathcal{M}$, defined by

$$\hat{A}\phi(x)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi(g(t)^{-1}x)\right)_{t=0} \tag{15}$$

where $x\in\mathcal{M}$,

$$x=(q^\mu,\eta^\alpha,\phi)\leftrightarrow\exp(q^\mu J_{\mu+}+\eta^\alpha L_{\alpha+})\exp(\phi J_{+-})G_0$$

represents the coset.

In a similar fashion, for $D(2,1;\alpha)$ we define the homogeneous manifold and stability group as

$$\begin{aligned}
&\mathcal{M}=D(2,1;\alpha)/\tilde{G}_0\\
&\tilde{G}_0=Osp(1,1/2)\wedge\mathcal{N}
\end{aligned}$$

where now $OSp(1,1/2) = \langle \tilde{J}, L_{\mu\alpha}, K_{\alpha\beta}\rangle$ and $\mathcal{N}$ is unchanged from (14). This leads to generators $\hat{A}$ defined as in (15), except now the coset representatives $x \in \mathcal{M}$ are given by

$$x \equiv (q^\mu, \eta^\alpha, \phi) \leftrightarrow \exp(q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+})\exp(\phi\tilde{J}_{+-})\tilde{G}_0.$$

The superfield realization for $D(2,1;\alpha)$ is computed in the standard way following (15). For example, it is clear that the even generators $J_{\mu\nu}$ associated with group elements $g(\epsilon) = \exp(\epsilon^\mu J_{\mu\nu})$ simply induce translations in the coordinates $q^\mu$, $\hat{J}_{\mu+} = -\partial/\partial q^\mu$. For later comparison, we re-scale the variables as follows:

$$p^\mu = \lambda^{-1}q^\mu \qquad \theta^\alpha = \lambda^{-1}\eta^\alpha \qquad \lambda = \mathrm{e}^\phi \quad (\lambda > 0).$$

Then, finally we have

$$J_{\mu+} = -\lambda^{-1}\frac{\partial}{\partial p^\mu}.$$

In appendix A we explicitly evaluate $J_{\mu+}$, as well as a further two generators, with the understanding that the remainder are done in a similar fashion.

The full set of generators for the superfield realization of $D(2,1;\alpha)$ is

$$\begin{aligned}
J_{\mu+} &= -\lambda^{-1}\frac{\partial}{\partial p^\mu}\\
L_{\alpha+} &= -\lambda^{-1}\frac{\partial}{\partial \theta^\alpha}\\
L_{\alpha-} &= \frac{1}{2}\lambda(p^\nu p_\nu + \theta^\beta\theta_\beta)\frac{\partial}{\partial\theta^\alpha} - \theta_\alpha\lambda^2\frac{\partial}{\partial\lambda} - a\lambda\theta_\alpha p^\mu\varepsilon_\mu{}^\nu\frac{\partial}{\partial p^\nu}\\
L_{\mu\alpha} &= p_\mu\frac{\partial}{\partial\theta^\alpha} - \theta_\alpha\frac{\partial}{\partial p^\mu} - a\theta_\alpha\varepsilon_\mu{}^\nu\frac{\partial}{\partial p^\nu}\\
K_{\alpha\beta} &= \theta_\alpha\frac{\partial}{\partial\theta^\beta} + \theta_\beta\frac{\partial}{\partial\theta^\alpha}\\
J &= -p^\mu\varepsilon_\mu{}^\nu\frac{\partial}{\partial p^\nu} + \frac{a}{1-a^2}\left(\lambda\frac{\partial}{\partial\lambda} - p^\mu\frac{\partial}{\partial p^\mu} - \theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right)\\
J_{+-} &= -\lambda\frac{\partial}{\partial\lambda} - \frac{a^2}{1-a^2}\left(\lambda\frac{\partial}{\partial\lambda} - p^\mu\frac{\partial}{\partial p^\mu} - \theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right)\\
J_{\mu-} &= \frac{1}{2}\lambda\theta^\alpha\theta_\alpha\frac{\partial}{\partial p^\mu} + \frac{1}{2}\lambda a\varepsilon_\mu{}^\nu\theta^\alpha\theta_\alpha\frac{\partial}{\partial p^\nu} + \frac{1}{2}\lambda\varepsilon_{\mu\nu}\varepsilon_\rho{}^\sigma p^\nu p^\rho\frac{\partial}{\partial p^\sigma}\\
&\quad -\lambda^2 p_\mu\frac{\partial}{\partial\lambda} + \frac{1}{2}\lambda p_\mu p^\nu\frac{\partial}{\partial p^\nu}\\
&\quad -\frac{\lambda a(ap_\mu + \varepsilon_{\mu\rho}p^\rho)}{1-a^2}\left(\lambda\frac{\partial}{\partial\lambda} - p^\nu\frac{\partial}{\partial p^\nu} - \theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right).
\end{aligned} \tag{16}$$

If we compare this realization with that obtained for $osp(d,2/2)$ (see [1]) the similarities are evident (although the realization of $J_{\mu-}$ requires some attention). In fact, if we allow $\alpha \to 1$ (and thus $a \to 0$), which corresponds to $D(2,1;\alpha) \cong osp(2,2/2)$, then it can easily be seen that the above relations are in fact identical to those obtained using the standard superfield [1] for the massless case. In appendix B the closure of these generators on the $D(2,1;\alpha)$ algebra is illustrated for the case of the anti-commutator $\{L_{\mu\alpha}, L_{\nu\beta}\} = \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta}$.

### 2.3. Physical states

The BRST operator for the $D(2,1;\alpha)$ model can be constructed by considering two linearly independent spinors $\eta^\alpha$ and $\eta'^\alpha$ which obey the condition $\eta^\alpha\eta'_\alpha = 1$, for example

$$\eta^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \eta'^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$$

We define the ghost number operator in the usual way,

$$N_{\rm gh} = \eta^\alpha \eta'^\beta K_{\alpha\beta} = (\eta \cdot \theta)(\eta' \cdot \partial) + (\eta' \cdot \partial)(\eta \cdot \partial) \tag{17}$$

where $\partial_\alpha = \partial/\partial\theta^\alpha$. Similarly we can define the BRST operator as

$$\begin{aligned}\Omega &= \eta^\alpha L_{\alpha-}\\ &= \eta^\alpha \left(\frac{1}{2}\lambda(p^\nu p_\nu + \theta^\beta\theta_\beta)\frac{\partial}{\partial\theta^\alpha} - \theta_\alpha\lambda^2\frac{\partial}{\partial\lambda} - a\lambda\theta_\alpha p^\mu \varepsilon_\mu{}^\nu \frac{\partial}{\partial p^\nu}\right).\end{aligned} \tag{18}$$

The physical states can be calculated by considering the effect of $\Omega$ on a superfield

$$\psi = A + \theta^\alpha\chi_\alpha + \tfrac{1}{2}\theta^\alpha\theta_\alpha B \tag{19}$$

(for more detail see [19]). Explicitly we can write

$$\begin{aligned}\Omega\psi = \frac{1}{2p_+}p\cdot p(\eta\cdot\chi) + \eta^\alpha\theta_\alpha\left(\frac{1}{2p_+}B + \left[\frac{\partial}{\partial p_+} - \frac{a}{p_+}(p^\mu\varepsilon_{\mu\nu}\partial_{p_\mu})\right]A\right)\\ +\frac{1}{2}\theta^\alpha\theta_\alpha\left(-\frac{\partial}{\partial p_+} + \frac{1}{p_+}(1 + ap^\mu\varepsilon_{\mu\nu}\partial_{p_\nu})\right)(\eta\cdot\chi)\end{aligned} \tag{20}$$

and impose the conditions

$$\Omega\psi = 0 \qquad \psi \neq \Omega\psi' \qquad \text{and} \qquad N_{\rm gh}\psi = \ell\psi$$

for the maximal eigenvalue $\ell = 1$ (in accordance with [19]).

Comparing equations (19) and (20) we see that the components $A$, $\chi_\alpha$ and $B$ of $\psi$ are defined up to addition of functions corresponding to coefficients in (20). Imposing the first condition above, we see that the $\eta^\alpha\theta_\alpha$ coefficient determines simply the $p_+$-dependence of $A$ in terms of some unknown $B$ (which is itself determined up to a $p_+$-derivative of some function).

Looking at $\eta^\alpha\chi_\alpha$, we find, respectively, from $\mathcal{O}(\theta^0)$ and $\mathcal{O}(\theta^2)$

$$\frac{1}{2p_+}(p\cdot p)(\eta^\alpha\chi_\alpha) = 0 \tag{21}$$

$$\left(\frac{\partial}{\partial p_+} - \frac{1 + ap^\mu\varepsilon_{\mu\nu}\partial^\nu}{p_+}\right)(\eta^\alpha\chi_\alpha) = 0. \tag{22}$$

We can write $p^\mu$ in component form as

$$p_R^\mu = \frac{1}{\sqrt{2}}(p^\mu + \varepsilon^\mu{}_\nu p^\nu)$$

$$p_L^\mu = \frac{1}{\sqrt{2}}(p^\mu - \varepsilon^\mu{}_\nu p^\nu).$$

Assuming that $\eta^\alpha\chi_\alpha = \phi(p)$, where $\phi(p)$ is a function of the form

$$\phi(p) = p_+\varphi\left((1 + a\ln p_+)p_R, (1 - a\ln p_+)p_L\right)$$

we have

$$\frac{\partial}{\partial p_+}\phi(p) = \frac{1}{p_+}\left(1 + a\left(p^\mu\varepsilon_\mu{}^\nu\frac{\partial}{\partial p^\nu}\right)\right)\phi$$

as in (22). Hence the given form of $\phi(p)$ solves equation (22).

Enforcing (21) gives that $\phi(p)$ satisfies $p\cdot p = 0$, or

$$\frac{1}{2p_+}p_R\cdot p_L p_+\varphi(\zeta^+ p_R, \zeta^- p_L) = 0$$

where $\zeta^{\pm} = 1 \pm a \ln p_+$. Equivalently, in Fourier space the constraints are solved by the physical states

$$\varphi(x_R, x_L) \equiv \int (\chi \cdot \psi) \mathrm{e}^{-\mathrm{i}p\cdot x}\, \mathrm{d}x$$

that satisfy

$$\frac{\partial}{\partial x_R}\frac{\partial}{\partial x_L}\varphi\left(\frac{x_R}{\zeta^+}, \frac{x_L}{\zeta^-}\right) = 0. \tag{23}$$

Moreover, if we assume $H\psi = 0$ ($H$ the Hamiltonian) on the physical states and we assume the Schrödinger equation

$$H = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}$$

then these functions are independent of $\tau$.

Finally, by once again employing the triviality of Im $\Omega$ for the maximal ghost number we can show that the physical state is unique, as there is no function $\psi'$ such that $\psi = \Omega\psi'$ lies in the same cohomology as $\psi$.

The system we have constructed can be interpreted as the 'quantization' of a classical '$D(2, 1; \alpha)$' particle. The $p_+$-dependence on physical wavefunctions provides indirect evidence that the model involves a more subtle implementation of world line diffeomorphisms than usual. Note that the two-dimensional case has the unique property that Lorentz invariance is not broken. The metric $x^\mu x^\nu \eta_{\mu\nu} = x_R x_L$ is still a world-line scalar if $x_R, x_L$ transform as densities:

$$x'_{R,L}(\tau')\, \mathrm{d}\tau'^{\pm a} = x_{R,L}(\tau)\, \mathrm{d}\tau^{\pm a}.$$

Corresponding covariant actions may be responsible (after gauge fixing) for the $p_+$-scaling behaviour[4].

## 3. Classical Hamiltonian and action

In the previous section we did not explicitly calculate the corresponding Hamiltonian $H = -\{\mathcal{F}, \Omega\}$, nor did we specify a gauge fixing function $\mathcal{F}$. The reason behind this is simple: as we have no classical model with which to compare our quantized particle we do not have any guide as to what our quantized Hamiltonian should look like, and thus no guide as to which gauge fixing function $\mathcal{F}$ we should choose. In this section we postulate an $\mathcal{F}$ which leads to an acceptable looking Hamiltonian, and from there derive a classical action $S$. This is the action which defines the classical system which corresponds to the quantum system derived from the algebraic structure in section 2.

By definition the gauge fixing fuction $\mathcal{F}$ is Grassmann odd and has ghost number $-1$, thus it obeys the equation

$$[\![N_{\mathrm{gh}}, \mathcal{F}]\!] = -\mathcal{F}. \tag{24}$$

As well as these constraints on $\mathcal{F}$, in the $D(2, 1; \alpha)$ system we must make sure that Hamiltonian generated is general enough to encompass the extended behaviour of the system (as compared with the corresponding $osp(2, 2/2)$ system) and that in the limit $\alpha \to 1$ it reduces to the Hamiltonian for the $osp(2, 2/2)$ system of [1].

Firstly we express the ghost number operator (see (17)) as

$$N_{\mathrm{gh}} = \tfrac{1}{2}(K_{22} - K_{11}) = \theta_2\partial_2 - \theta_1\partial_1.$$

[4] The gauge equivalence class of $\lambda$, or $e$, namely $\int_{\tau_i}^{\tau_f} e(\tau)\, \mathrm{d}\tau$, is proportional to $\lambda$ in the present case $\dot{\lambda} = 0$.

As a first guess at the gauge fixing function we choose $\mathcal{F} = \eta'^{\alpha}\theta_\alpha$ (where $\eta'$ is one of the spinors introduced in the previous section). This function has ghost number −1 and is Grassmann odd; checking that it satisfies (24) we get

$$[\![N_{\rm gh}, \mathcal{F}]\!] = \frac{1}{\sqrt{2}}\{\theta_2\partial_2 - \theta_1\partial_1, \theta_2 - \theta_1\} = -\frac{1}{\sqrt{2}}(\theta_2 - \theta_1) = -\mathcal{F}$$

as we desire. However this $\mathcal{F}$ falls down when we generate the corresponding Hamiltonian, which is found to be independent of $\alpha$. Thus the Hamiltonian generated by this gauge fixing function cannot reproduce the $\alpha$-dependent quantized system of section 2.

Our second choice for the gauge fixing function is $\mathcal{F} = \eta'^{\alpha}\partial_\alpha$. This function also satisfies the necessary conditions; however, it once again falls down when we generate the Hamiltonian, as this $H$ does not revert to the $osp(2, 2/2)$ Hamiltonian as $\alpha \to 1$. Thus we are led to choosing our gauge fixing function as a scalar combination of the two given above, i.e.

$$\mathcal{F} = \frac{1}{\sqrt{2}}\left((\theta_2 - \theta_1) + b(\partial_1 - \partial_2)\right) \tag{25}$$

where $b$ is an arbitrary scalar, and we have changed the overall sign of the second term. This $\mathcal{F}$ is Grassmann odd, has ghost number −1 and obeys equation (24). The corresponding Hamiltonian can now be calculated:

$$\begin{aligned} H &= [\![\Omega, \mathcal{F}]\!] = [\![\Omega, \mathcal{F}_\theta - b\mathcal{F}_\partial]\!] \\ [\![\Omega, \mathcal{F}_\theta]\!] &= \left[\!\left[\tfrac{1}{2}\lambda p^2(\partial_1 + \partial_2) + \tfrac{1}{2}\lambda\theta^\beta\theta_\beta(\partial_1 + \partial_2), \theta_2 - \theta_1\right]\!\right] \\ &= -\tfrac{1}{2}\lambda(p^2 + \theta^\beta\theta_\beta) \end{aligned}$$

and

$$\begin{aligned} [\![\Omega, -\mathcal{F}_\partial]\!] &= \tfrac{1}{2}\left[\!\left[ - (\theta_1 + \theta_2)\lambda^2\partial_\lambda - a\lambda(\theta_1 + \theta_2)p^\mu\varepsilon_\mu^{\ \nu}\partial_\nu + \lambda\theta^\beta\theta_\beta(\partial_1 + \partial_2), \partial_a - \partial_2\right]\!\right] \\ &= \lambda^2\partial_\lambda + a\lambda p^\mu\varepsilon_\mu^{\ \nu}\partial_\nu + \tfrac{1}{2}\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2). \end{aligned}$$

Thus the Hamiltonian for the $D(2, 1; \alpha)$ system is

$$H = -\tfrac{1}{2}\lambda\left(p^2 + \theta^\beta\theta_\beta\right) + b\lambda^2\partial_\lambda + ab\lambda p^\mu\varepsilon_\mu^{\ \nu}\partial_\nu + \tfrac{1}{2}b\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2) \tag{26}$$

where $\lambda = 1/p_+$. Notably this action is general enough so as to encompass the special behaviour of the $D(2, 1; \alpha)$ system (as $a = a(\alpha)$) and can reduce to the Hamiltonian for the massless scalar particle in the $osp(2, 2/2)$ case of [1].

Having derived the Hamiltonian of the $D(2, 1; \alpha)$ system we now seek to calculate the corresponding classical action and Lagrangian $\mathcal{L}$. We do this by means of a Legendre transformation and the Hamiltonian equations of motion. Given the Hamiltonian, we can write the Lagrangian as

$$\mathcal{L} = \sum \dot{q}p - H(q, p) \tag{27}$$

where $q$, $p$ are the generalized coordinates of $H$. $\dot{q}$ is calculated by means of the Hamiltonian equations of motion:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} = \lambda(-p^\mu + ab\varepsilon^\mu_{\ \nu}x^\nu) \\ \dot{\lambda} &= \frac{\partial H}{\partial\partial_\lambda} = b\lambda^2 \\ \dot{\theta^\alpha} &= \frac{\partial H}{\partial\partial_\alpha} = \frac{1}{\sqrt{2}}\lambda b(\theta_1 - \theta_2)\eta^\alpha\partial_\alpha \\ \therefore\quad \dot{\theta}^\alpha\partial_\alpha &= \tfrac{1}{2}\lambda b(\theta_1 - \theta_2)(\partial_1 + \partial_2). \end{aligned} \tag{28}$$

Note that here we have used

$$\eta^\alpha \partial_\alpha = \frac{1}{\sqrt{2}}(\partial_1 + \partial_2).$$

From the first of these equations we can write

$$p_\mu = -\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu$$

and so, by substituting the above expressions into (27), we get

$$\begin{aligned}\mathcal{L} = \dot{x}^\mu \left(-\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu\right) + b\lambda^2\partial_\lambda + \frac{1}{2}\lambda b(\theta_1 - \theta_2)(\partial_1 + \partial_2) \\ +\frac{1}{2}\lambda\left(\left(\frac{-\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu\right)\left(\frac{-\dot{x}^\mu}{\lambda} - ab\varepsilon^{\mu\nu}x_\nu\right) + \theta^\beta\theta_\beta\right) - b\lambda^2\partial_\lambda \\ -ab\lambda\left(\frac{\dot{x}^\mu}{\lambda} - ab\varepsilon^\mu_\nu x^\nu\right)\varepsilon^\nu_\mu x_\nu - \frac{1}{2}b\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2).\end{aligned} \tag{29}$$

Thus the Lagrangian can be written

$$\mathcal{L} = -\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \lambda\frac{1}{2}(ab)^2x^2 - ab\varepsilon_{\mu\nu}\dot{x}^\mu x^\nu + \frac{1}{2}\lambda\theta^\beta\theta_\beta. \tag{30}$$

The classical Lagrangian (and action) corresponding to the quantum Hamiltonian (26) should be free of ghosts (as they only arise in the extended phase space of the BFV–BRST construction). Likewise the canonical momentum conjugate to the Lagrange multiplier $\lambda$ should not be present. Thus we arrive at the classical action of the $D(2,1;\alpha)$ system by decoupling the ghost sector from the action above, i.e. only considering the bosonic part

$$S = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left[-\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \lambda\frac{1}{2}(ab)^2x^2 - ab\varepsilon_{\mu\nu}\dot{x}^\mu x^\nu\right]. \tag{31}$$

By comparing this with the action given in [1] we can see that (31) corresponds to a massless scalar particle in a potential well. In fact if we ignore the last term in (31) then we have arrived at the classical action of an oscillating massless particle (i.e. where the potential is proportional to $x^2$). For further details of the oscillator in the classical or quantum case see [20, 21]. The final term of (31) introduces a cross term between velocity and position. Comparing this term with the potential term in [1], we see that the cross term is similar to that produced by a homogeneous electromagnetic field $F_{\mu\nu} = ab\varepsilon_{\mu\nu}$.

The action (31) also satisfies the condition that as $a \to 0$ (which is equivalent to $\alpha \to 1$), $\mathcal{L}$ becomes the Lagrangian of the massless scalar particle. The constant $b$ is also important as it distinguishes between the parts of the action that arise from each of the two gauge fixing functions we tried earlier; $\mathcal{F}_\theta$ and $\mathcal{F}_\partial$. We can now set $b = 1$ without affecting the behaviour of the particle.

For the sake of completeness, we shall identify the total covariant energy and angular momentum of the classical $D(2,1;\alpha)$ particle, as well as calculating the Euler–Lagrange equations of motion. The total covariant energy is given by

$$\begin{aligned}P_\mu &= \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \\ &= -\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu.\end{aligned}$$

The total angular momentum is

$$\begin{aligned}M_{\mu\nu} &= P_\mu x_\nu - P_\nu x_\mu \\ &= \frac{1}{\lambda}(\dot{x}_\nu x_\mu - \dot{x}_\mu x_\nu) + ab(\varepsilon_{\nu\rho}x_\mu - \varepsilon_{\mu\rho}x_\nu)x^\rho.\end{aligned} \tag{32}$$

The Euler–Lagrange equations of motion are

$$-\frac{\ddot{x}_\mu}{\lambda} + (ab)^2\lambda x_\mu - 2ab\varepsilon_{\mu\nu}\dot{x}^\nu = 0 \tag{33}$$

$$\frac{\dot{x}^2}{2\lambda^2} - (ab)^2 x^2 = 0. \tag{34}$$

By virtue of the fact that the Lagrangian is independent of $\dot{\lambda}$, the second of these two equations is actually a constraint on the system. Hence, the Hessian of the Lagrange function vanishes identically, except for its components $\partial^2\mathcal{L}/(\partial\dot{x}^\mu\partial\dot{x}^\nu)$ [12], thereby illustrating the singular nature of the action (31). Thus we can identify (34) as a first-class constraint of the $D(2, 1; \alpha)$ system.

Now that we have determined the classical action corresponding to the $D(2, 1; \alpha)$ particle it is possible to start the loop again, so to speak: that is, identify the first class constraints, extend the phase space and follow the BFV–BRST quantization procedure in order to arrive at the quantized $D(2, 1; \alpha)$ particle. However, we shall not do this; given that we correctly choose the gauge fixing condition for the scalar particle condition we would end up with exactly the same quantized system as system as that obtained in section 2. Secondly, the aim in this paper is to study the algebras of quantization, whch we have already done for the $D(2, 1; \alpha)$ particle in the previous section.

## 4. Conclusion

In this paper we have shown that it is possible to extend the BFV–BRST quantization algebra $iosp(d, 2/2)$ in two dimensions into the more general classical simple Lie superalgebra $D(2, 1; \alpha)$. To do this we started without a classical physical model of a particle, and so relied entirely on the algebraic structure as our guide. In section 2 we showed that the algebraic model that we had constructed was an admissible quantization superalgebra, and so provided a quantization of the corresponding classical system. In section 3 we then calculated the classical action corresponding to the quantum system. If this action were used as a starting point, then the BFV–BRST process could be followed and the quantum system of equation (2) would be derived.

An alternative (and equally valid) method of presenting this paper would have been to start with the classical action (31) and from there proceed to quantize the system and demonstrate that it obeys a $D(2, 1; \alpha)$ quantization superalgebra.

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## Appendix A. Calculations for superfield realization

In this appendix we explicitly calculate the superfield realization for $D(2, 1; \alpha)$ by introducing formal group elements $g = \mathrm{e}^{\phi\cdot F}$ for the generators $F_1, F_2, \ldots, F_N$ and graded parameters $\phi_1, \phi_2, \ldots, \phi_N$ and evaluate the product

$$h \cdot g = \mathrm{e}^{\epsilon\cdot F}\mathrm{e}^{\phi\cdot F}$$

in order to find the product map $\mu(\epsilon, \phi)$ to first order in $\epsilon$.

Thus we can calculate $J_{\mu+}$ as follows:

Let $\epsilon = \mathrm{e}^{\epsilon^\mu J_{\mu+}}$, then

$$\begin{aligned}\epsilon^{-1} \cdot x &= \mathrm{e}^{-\epsilon^\mu J_{\mu+}} \mathrm{e}^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0 \\ &= \mathrm{e}^{(q^\mu - \epsilon^\mu) J_{\mu+} + \eta^\alpha L_{\alpha+}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0.\end{aligned}$$

So if we consider functions $f(q^\mu, \eta^\alpha, \phi)$, then

$$\epsilon f(q^\mu, \eta^\alpha, \phi) = f(q - \epsilon, \eta, \phi) \tag{A.1}$$

and so

$$\delta f = \epsilon f - f = -\epsilon^\mu \frac{\partial}{\partial q^\mu} f(q^\mu, \eta^\alpha, \phi) + \cdots.$$

Hence from (15) we can write

$$J_{\mu+} = -\frac{\partial}{\partial q^\mu}. \tag{A.2}$$

We shall explicitly calculate a further two generators, with the understanding that the remainder can be derived in a similar fashion.

Let $\xi = \mathrm{e}^{\xi^\alpha L_{\alpha+}}$, thus

$$\begin{aligned}\xi^{-1} x &= \mathrm{e}^{-\xi^\alpha L_{\alpha-}} \mathrm{e}^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0 \\ &= \mathrm{e}^{q^\mu J_{\mu+}} \mathrm{e}^{\eta^\alpha L_{\alpha+} - \xi^\alpha L_{\alpha+}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0.\end{aligned}$$

Therefore we can see that

$$\delta f = -\xi^\alpha \frac{\partial}{\partial \eta^\alpha} f + \cdots$$

and so we have the realization

$$L_{\alpha+} = -\frac{\partial}{\partial \eta^\alpha}. \tag{A.3}$$

Once again, scaling variables gives us $L_{\alpha+}$ in the momentum representation

$$L_{\alpha+} = -\lambda^{-1} \frac{\partial}{\partial \theta^\alpha}.$$

Lastly, we shall calculate $L_{\mu\alpha}$; let $\rho = \mathrm{e}^{\rho^{\mu\alpha} L_{\mu\alpha}}$, and so we have

$$\begin{aligned}\rho^{-1} x &= \mathrm{e}^{-\rho^{\mu\alpha} L_{\mu\alpha}} \mathrm{e}^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0 \\ &= \mathrm{e}^{q^\mu J_{\mu+} - [\rho^{\nu\alpha} L_{\nu\alpha}, q^\mu J_{\mu+}]} \mathrm{e}^{\eta^\alpha L_{\alpha+} - [\rho^{\nu\alpha} L_{\nu\alpha}, \eta^\beta L_{\beta+}]} \mathrm{e}^{\rho^{\mu\alpha} L_{\mu\alpha}} \mathrm{e}^{\phi \tilde{J}_{+-}} G_0.\end{aligned}$$

To simplify this expression we use the commutation relations

$$\begin{aligned} q^\mu \rho^{\nu\alpha} [J_{\mu+}, L_{\nu\alpha}] &= \rho^{\nu\alpha} q_\nu L_{\alpha+} \\ -\eta^\beta \rho^{\nu\alpha} \{L_{\nu\alpha}, L_{\beta+}\} &= \eta^\beta \rho^{\nu\alpha} \varepsilon_{\alpha\beta} (J_{\nu+} + a \varepsilon_\nu^\mu J_{\mu+}) \\ &= -\rho^{\nu\alpha} \eta_\alpha J_{\nu+} - a \rho^{\mu\alpha} \eta_\alpha \varepsilon_\mu^\nu J_{\mu+}.\end{aligned}$$

Thus we can write

$$\rho f(q, \eta, \phi) = f(q^\mu - \rho^{\mu\alpha} \eta_\alpha - \rho^{\nu\alpha} a \eta_\alpha \varepsilon_\nu^\mu, \eta^\alpha + \rho^{\nu\alpha} q_\nu, \phi)$$

and so

$$L_{\nu\alpha} = -\eta_\alpha \left( \frac{\partial}{\partial q^\nu} + a \varepsilon_\nu^\mu \frac{\partial}{\partial q^\mu} \right) + q_\nu \frac{\partial}{\partial \eta^\alpha}. \tag{A.4}$$

Once again, changing to momentum representation and re-arranging gives us

$$L_{\mu\alpha} = p_\mu \frac{\partial}{\partial \theta^\alpha} - \theta_\alpha \frac{\partial}{\partial p^\mu} - a \theta_\alpha \varepsilon_\mu{}^\nu \frac{\partial}{\partial p^\nu}.$$

## Appendix B. Closure of algebra generated by $J_{MN}$

Although the commutation relations of the generators $J_{MN}$ given in (16) must be equal to those given in (12) and (6), we shall test this by calculating $\{L_{\mu\alpha}, L_{\nu\beta}\}$ (we could choose any of the relations).

Equation (12) tells us that $\{L_{\mu\alpha}, L_{\nu\beta}\} = \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta}$. Using the definition of $L_{\mu\alpha}$ given in (A.4) gives

$$\{-\eta_\alpha\partial_\mu - a\eta_\alpha\varepsilon^{\nu}_{\mu}\partial_\sigma + q_\mu\partial_\alpha, -\eta_\beta\partial_\nu - a\eta_\beta\varepsilon^{\rho}_{\nu}\partial_\rho + q_\nu\partial_\beta\}$$
$$= -\{\eta_\alpha\partial_\mu, q_\nu\partial_\beta\} - a\{\eta_\alpha\varepsilon^{\sigma}_{\mu}\partial_\sigma, q_\nu\partial_\beta\} - \{q_\mu\partial_\alpha, \eta_\beta\partial_\nu\} - a\{q_\mu\partial_\alpha, \eta_\beta\varepsilon^{\rho}_{\nu}\partial_\rho\}.$$

Note that the first and third terms are identical, except for the indices, as are the second and fourth terms. Using the identity

$$\{AB, CD\} = \tfrac{1}{2}\{A, C\}\{B, D\} + \tfrac{1}{2}[A, C][B, D]$$

where $[A, B] = [C, D] = 0$, we have that the first term is

$$-\tfrac{1}{2}\eta_{\mu\nu}(2\eta_\alpha\partial_\beta - \varepsilon_{\alpha\beta}) - \tfrac{1}{2}\varepsilon_{\alpha\beta}(2q_\nu\partial_\mu + \eta_{\mu\nu}).$$

So terms one and three sum to

$$\varepsilon_{\alpha\beta}(q_\mu\partial_\nu - q_\nu\partial_\mu) - \eta_{\mu\nu}(\eta_\alpha\partial_\beta + \eta_\beta\partial_\alpha).$$

In a similar fashion, we get that the second and fourth terms sum to

$$-a\varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\eta^{\gamma}\partial_\gamma - a\varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}q^{\rho}\partial_\rho.$$

Combining these two expressions together we get

$$\begin{aligned}\{L_{\mu\alpha}, L_{\nu\beta}\} &= \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\left(-q^{\rho}\varepsilon^{\sigma}_{\rho}\partial_\sigma - aq^{\rho}\partial_\rho - a\eta^{\gamma}\partial_\gamma\right) - \eta_{\mu\nu}\left(\eta_\alpha\partial_\beta + \eta_\beta\partial_\alpha\right)\\ &= \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta} \qquad \text{(B.1)}\end{aligned}$$

as claimed.

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